THERMODYNAMIC APPROACH TO THE EQUATION OF STATE OF A MAGNETIZED FERMI GAS

V. CANUTO*, H. Y. CHIU**

Institute for Space Studies, Goddard Space Flight Center, NASA, New York, N.Y., U.S.A.

and

L. FASSIO-CANUTO

Dept. of Physics, Centro de Investigacion y de Estudios Avanzados, Mexico 14, D.F., Mexico

(Received 2 September, 1968)

Abstract. In this paper we have rederived the equations of state for a magnetized Fermi gas by generalizing the physical definition of the pressure. We have also given a simplified derivation of the energy eigenvalues of a free electron in a magnetic field, based on the use of simple harmonic oscillators. Physical interpretations of our results are presented. Possible astrophysical applications are also discussed.

1. Introduction

According to classical electrodynamics the magnetic field of a current carrying plasma increases as R^{-2}, where R is a scale factor. If we apply this criteria to the gravitational collapse of a star which in its main sequence possesses a moderate field of a few thousand gauss, then the final field strength is of the order of 10^{14} gauss, assuming that the initial radius is of the order 10^{11} cm and the final radius after collapse (neutron star) is of the order of 10^{6} cm. Classical electrodynamics will break down in such a strong field as the spin magnetic-field interaction energy is of the order of mc^2 when the field strength exceeds 10^{14} gauss.

In a series of papers (Canuto and Chiu, 1968a, b, c; Chiu and Canuto, 1968) we have derived the equations of state of a relativistic electron gas in an intense external magnetic field (a magnetized Fermi gas). Our calculations were based on the use of energy-momentum tensor $T_{\mu\nu}$, for which the exact eigenfunctions of the Dirac equation were used. However, the use of quantum electrodynamics in obtaining thermodynamic functions, though intrinsically fundamental, does not offer a direct insight into how the equation of state differs from that of an ordinary Fermi gas in the absence of a magnetic field. It is the purpose of this paper to rederive these equations of state, by analogy with the more physical definition of thermodynamic functions, such as the pressure and the internal energy. This paper thus does not contain new results, but gives a more physical interpretation of our previous results which contain the formal use of field-theoretical techniques based on quantum electrodynamics.

* On leave of absence from Dept. of Physics, C.I.E.A., Ap. Post. 14740, Mexico 14, D.F., Mexico.
** Also with Physics Dept. and Earth and Space Sciences Dept., State University of New York, Stony Brook, N.Y., U.S.A.

Astrophysics and Space Science **3** (1969) 258–267; © *D. Reidel Publishing Company, Dordrecht-Holland*

2. Properties of the Eigenvalues of a Free Electron in an External Magnetic Field. The Critical Field

Let us first consider a free, non-relativistic electron of charge $e(e>0)$ in an external magnetic field H described by a vector potential $\mathbf{A}(x, y, z)$, $[\mathbf{H}=\operatorname{curl}\mathbf{A}]$. The Hamiltonian is given by the well-known expression (LANDAU and LIFSCHITZ, 1958)

$$\mathscr{H} = \frac{1}{2m}(\mathbf{p} - (e/c)\,\mathbf{A})^2 . \tag{1}$$

If the magnetic field is constant in time and uniform in space, the vector potential $\mathbf{A}$ is given by:

$$\mathbf{A} = \tfrac{1}{2}\mathbf{H} \times \mathbf{r} . \tag{2}$$

Consider the case in which H is along the z-axis, i.e., $H_x = H_y = 0$, and $H_z = H$. Then Equation (2) gives:

$$A_x = -\tfrac{1}{2}yH; \qquad A_y = \tfrac{1}{2}xH; \qquad A_z = 0 . \tag{3}$$

With Equation (3) the non-relativistic Hamiltonian $\mathscr{H}$ can be put into the following form:

$$\mathscr{H} = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2 , \tag{4}$$

where

$$v_x = \frac{1}{m}\left(p_x - \frac{eH}{2c}\,y\right); \qquad v_y = \frac{1}{m}\left(p_y + \frac{eH}{2c}\,x\right); \qquad v_z = \frac{1}{m}\,p_z . \tag{5}$$

v_x, v_y and v_z are the generalized velocities of a free electron in an external magnetic field.

We now calculate the commutation relation between v_x and v_y. Remembering that

$$\left.\begin{aligned} [x, y] &= [p_x, p_y] = 0 , \\ [p_x, x] &= [p_y, y] = \hbar/i \end{aligned}\right\} \tag{6}$$

we easily obtain:

$$[v_x, v_y] = \frac{\hbar}{i}\,\frac{eH}{m^2 c} , \tag{7}$$

which vanishes for the case $H=0$, as expected. The effect of a magnetic field is to cause the velocities of a free electron to become non-commutative. Physically this means that both v_x and v_y cannot be measured simultaneously to arbitrary accuracy. Classically the trajectories of an electron with $v_z=0$ in a magnetic field are circles for which the two components of the velocity perpendicular to the field are correlated. A measurement of one component (say, v_x) will cause the circle to be displaced, thus altering the other component of the velocity (say, v_y).

Multiplying both sides of Equation (7) by $m(\hbar/mc^2)$, the left-hand side becomes the commutator between two quantities with the dimensions of a magnetic-field independent coordinate $\xi_x \equiv v_x/(\hbar/mc^2)$ and a momentum $\Pi_y \equiv mv_y$. We obtain:

$$[v_x, v_y]\, m \frac{\hbar}{mc^2} = [\xi_x, \Pi_y] = \frac{\hbar}{i} \frac{eH}{m^2 c}\, m \frac{\hbar}{mc^2} \equiv \frac{\hbar}{i} \frac{H}{H_c}, \tag{8}$$

whereas we have introduced the 'critical magnetic field' H_c given by

$$H_c = \frac{m^2 c^3}{e\hbar} = 4.414 \times 10^{13} \text{ gauss}. \tag{9}$$

Equation (8) implies that when $H > H_c$, ordinary non-relativistic quantum mechanics is violated. This is illustrated in the following two examples.

First, when $H > H_c$, the classical radius of gyration of an electron, r_L (Larmor radius) which is given by equating the centrifugal force mv^2/r_L to the magnetic force evH/c, i.e.

$$r_L = \frac{mcv}{eH} = \frac{\hbar}{mc} \frac{H_c}{H} \equiv \lambda\!\!\!^{-}_c \frac{H_c}{H}, \qquad (\lambda\!\!\!^{-}_c = \hbar/mc) \tag{10}$$

becomes less than the Compton wavelength of the particle. This is not allowed in ordinary relativistic quantum mechanics since no particles can be confined to a distance less than their Compton wavelengths, otherwise *Zitterbewegung* phenomenon has to be taken into account (Sakurai, 1967; Newton and Wigner, 1949).

As a second example let us consider the magnetic moment of the electron due to its circular motion around the magnetic field. We have

$$\mu = \pi r_L^2 \frac{ev}{2\pi c r_L}, \tag{11}$$

which is the product of the area of the circular orbit $\pi r_L{}^2$ times the current. Using Equation (10), we have

$$\mu = \mu_B \frac{H_c}{H}, \tag{12}$$

where $\mu_B = e\hbar/2mc$ is the Bohr magneton. Again, when $H > H_c$, $\mu < \mu_B$ which is not allowed, since the minimum value of the ratio μ/μ_B is expected to be unity.

We shall now obtain the energy eigenvalues. Let us introduce the following variables (Landau, 1930):

$$P = v_x m^{1/2}; \qquad Q = \frac{cm^{3/2}}{eH} v_y, \tag{13}$$

with which Equation (7) takes the form:

$$[P, Q] = \hbar/i.$$

If we regard P and Q as the new momentum and coordinate of the system, then P and Q satisfy the usual commutation relation (6). Using Equation (13), the nonrelativistic

Hamiltonian given in Equation (4) becomes:

$$\mathscr{H} = \tfrac{1}{2}P^2 + \tfrac{1}{2}(eH/mc)^2\, Q^2 + \tfrac{1}{2}mv_z^2\,, \tag{14}$$

which represents a two-dimensional harmonic oscillator in the x-y plane with free motion in the z-direction. The frequency is eH/mc, and the energy eigenvalues are known to be:

$$\left.\begin{aligned} E &= \hbar\omega(n+\tfrac{1}{2}) + \frac{1}{2m}p_z^2 = \frac{e\hbar_{\mathrm{H}}}{mc}(n+\tfrac{1}{2}) + \frac{1}{2m}p_z^2 \\ &= mc^2\frac{H}{H_{\mathrm{c}}}(n+\tfrac{1}{2}) + \frac{1}{2m}p_z^2\,. \end{aligned}\right\} \tag{15}$$

So far we have neglected the spin. The spin can be included in Equation (15) by adding to Equation (1) the so-called Pauli term (Sakurai, 1967):

$$\mu_{\mathrm{B}}\boldsymbol{\sigma}\cdot\mathbf{H}\,, \tag{16}$$

where $\boldsymbol{\sigma}$ is the spin operator, giving an additional term to the energy eigenvalues:

$$\frac{1}{2}\frac{e\hbar}{mc}Hs = \frac{1}{2}mc^2\frac{H}{H_{\mathrm{c}}}s\,, \qquad s = \pm 1\,, \tag{17}$$

which, when added to Equation (15), gives the final result as:

$$E(p_z, n, s) = \frac{p_z^2}{2m} + \tfrac{1}{2}mc^2\frac{H}{H_{\mathrm{c}}}(2n+1+s)\,. \tag{18}$$

The obvious generalization of Equation (18) to the relativistic case gives

$$E(p_z, n, s) = \left\{m^2c^4 + c^2p_z^2 + m^2c^4\frac{H}{H_{\mathrm{c}}}(2n+1+s)\right\}^{1/2}, \tag{19}$$

which is the correct expression for the energy eigenvalues of the Dirac equation for a free electron in an external magnetic field H, discussed in Canuto and Chiu (1968a), where references to original works can be found.

3. The Equations of State

A usual physical picture of the pressure is the force derived from the exchange of momenta of gas particles with the wall of a container in which the gas is confined. In Canuto and Chiu (1968a) we calculated the energy-momentum tensor for an electron gas in an external static and uniform magnetic field:

$$T_{\mu\nu} = \tfrac{1}{2}\hbar c\left[\bar{\psi}\gamma_\mu\delta_\nu\psi - \delta_\nu\bar{\psi}\gamma_\mu\psi\right] \tag{20}$$

using the exact wave-function solutions of the Dirac equation. As is well known, the diagonal components of the energy-momentum tensor, T_{xx}, T_{yy}, and T_{zz} are just pressures in x-, y-, and z-directions (Landau and Lifshitz, 1962). We shall now show that the usual physical picture of the pressure is preserved even in the presence of a magnetic field, provided that a reinterpretation and generalization of the velocity is used.

Because of the cylindrical symmetry of the problem, $T_{xx}=T_{yy}$. Let $P_{\perp})$ be the pressure in the direction perpendicular to the field and let $P_{\perp}$ be the pressure in the direction parallel to the field. We then have

$$P_{\perp} = T_{xx} = T_{yy}\,,$$
$$P_{\parallel} = T_{zz}\,.$$

From elementary thermodynamic considerations we have (LANDAU and LIFSHITZ, 1962):

$$\left.\begin{aligned} P_{\perp} &= \langle m v_x \cdot v_x \rangle = \langle m v_y \cdot v_y \rangle \\ P_{\parallel} &= \langle m v_z \cdot v_z \rangle\,, \end{aligned}\right\} \tag{21}$$

where the symbol $\langle \ldots \rangle$ stands for an average over a statistical ensemble. If the system is isotropic (no magnetic field) it is obvious that $P_{\perp}=P_{\parallel}$. Because of the presence of a magnetic field, the system is now anisotropic and therefore $P_{\perp} \neq P_{\parallel}$. However, it is easily seen that in the absence of interaction, the spatial components of the non-diagonal elements of the energy-momentum tensor vanish.

Consider now the quantity mv_x^2. Its generalization to include relativistic effects is made through the simple relations:

$$m v_x^2 \to \frac{m v_x^2}{\sqrt{1-\beta^2}} = \frac{mc^2}{E}\,\frac{m v_x^2 c^2}{c^2(1-\beta^2)} = \frac{c^2}{E}(mcu_x)^2 = \frac{c^2 p_x}{E}\,, \tag{22}$$

where $\beta = v/c$ and

$$p_\mu = mcu_\mu\,, \qquad u_x = v_x/\sqrt{1-\beta^2}\,, \qquad u_t = 1/\sqrt{1-\beta^2}\,,$$
$$\mu = x,\, y,\, z,\, t\,.$$

p_μ is the momentum four-vector (LANDAU and LIFSHITZ, 1962). Analogously

$$m v_y^2 \to \frac{c^2 p_y^2}{E}\,, \qquad m v_z^2 \to \frac{c^2 p_z^2}{E}\,. \tag{23}$$

Comparing now Equation (19) with the corresponding one for the case of $H=0$, i.e.,

$$E = \sqrt{m^2 c^4 + c^2 p_x^2 + c^2 p_y^2 + c^2 p_z^2} \tag{24}$$

we see that in the presence of a magnetic field,

$$c^2 p_x^2 + c^2 p_y^2 \to m^2 c^4 \frac{H}{H_c}(2n+1+s)\,; \qquad cp_z \to cp_z\,. \tag{25}$$

We therefore have (see also Equation (21)):

$$P_{\perp} = \langle m v_x^2 \rangle \to \left\langle \frac{c^2 p_x^2}{E} \right\rangle = \tfrac{1}{2} m^2 c^4 \frac{H}{H_c} \left\langle \frac{2n+1+s}{E(p_z, n, s)} \right\rangle, \tag{26}$$

$$P_{\parallel} = \langle m v_z^2 \rangle \to \left\langle \frac{c^2 p_z^2}{E} \right\rangle = \left\langle \frac{c^2 p_z^2}{E(p_z, n, s)} \right\rangle. \tag{27}$$

The statistical average, as indicated by the symbol $\langle \ldots \rangle$, is achieved by multiplying the quantity of interest by the Fermi distribution function and summing over all states. The Fermi distribution function $F(p_z, n, s)$, is given by

$$F(p_z, n, s) = \{1 + \exp \tilde{\beta}[E(p_z, n, s) - \tilde{\mu}]\}^{-1} \tag{28}$$

where $\tilde{\beta} = (kT)^{-1}$, T is the temperature and $\tilde{\mu}$ is the chemical potential. We thus obtain the expressions for $P_\perp$ and $P_\parallel$ as follows:

$$P_\perp = \tfrac{1}{2} mc^2 \frac{H}{H_c} \sum_{s=-1}^{+1} \sum_{n=0}^{\infty} \omega_n \sum_{p_z} \frac{(2n + 1 + s)\, mc^2}{E(p_z, n, s)} F(p_z, n, s) \tag{29}$$

$$P_\parallel = \sum_{s=-1}^{+1} \sum_{n=0}^{\infty} \omega_n \sum_{p_z} \frac{c^2 p_z^2}{E(p_z, n, s)} F(p_z, n, s). \tag{30}$$

ω_n is the degeneracy of the level n. The summation over p_z is easily transformed into an integration over p_z through the usual relation:

$$\sum_{p_z} \rightarrow \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}p_z = \frac{1}{2\pi\lambda_c} \int_{-\infty}^{\infty} \mathrm{d}\left(\frac{p_z}{mc}\right). \tag{31}$$

The calculation of ω_n is a little more involved and can be done in the following way (KUBO, 1965). When $H=0$, the number of levels in $\mathrm{d}p_x \mathrm{d}p_y$ at p_x and p_y is given by

$$\frac{\mathrm{d}p_x \mathrm{d}p_y}{(2\pi\hbar)^2}. \tag{32}$$

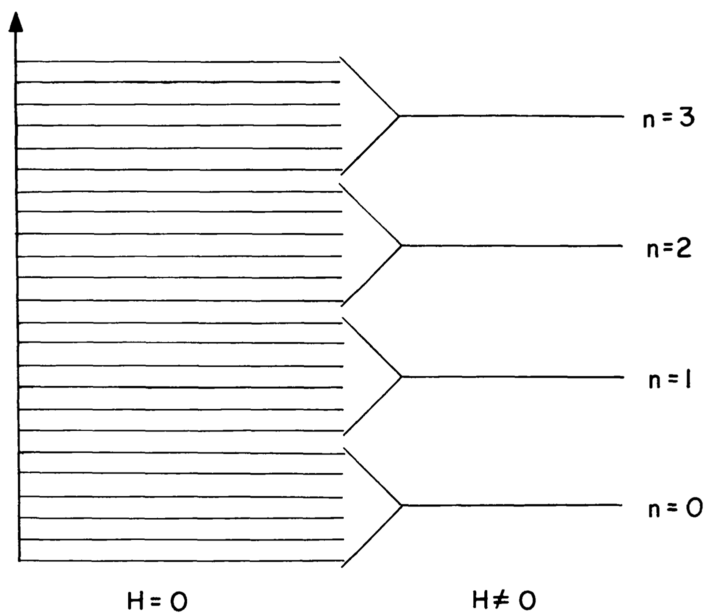

Fig. 1. The coalescence of free particle states into equally spaced harmonic oscillator energy states in the presence of a magnetic field.

In the presence of H these levels coalesce into those of a harmonic oscillator, as shown in Figure 1. The degeneracy of each of these levels is therefore given by integrating Equation (32) as follows:

$$\omega_n = \frac{1}{(2\pi\hbar)^2} \int\limits_{A < \frac{p_x{}^2 + p_y{}^2}{2m} < B} \mathrm{d}p_x \, \mathrm{d}p_y \,, \tag{33}$$

where (cf. Equation (15))

$$A = mc^2 \frac{H}{H_c} n \,; \qquad B = mc^2 \frac{H}{H_c} (n+1) \,. \tag{34}$$

Introducing the cylindrical coordinates (p, ϕ):

$$p_x = p \cos\phi \,; \quad p_y = p \sin\phi \,; \quad \mathrm{d}p_x \, \mathrm{d}p_y = p \, \mathrm{d}p \, \mathrm{d}\phi \,,$$

we obtain:

$$\omega_n = \frac{1}{(2\pi\hbar)^2} \int\limits_0^{2\pi} \mathrm{d}\phi \int\limits_{A < \frac{p^2}{2m} < B} p \, \mathrm{d}p = \frac{2\pi}{(2\pi\hbar)^2} \frac{1}{2} 2m (B - A)$$

or finally

$$\omega_n = \frac{1}{2\pi} \frac{1}{\lambda_c^2} \frac{H}{H_c} \,. \tag{35}$$

Substituting Equation (31) and (35) into Equations (29)–(30), we obtain:

$$P_\perp = \frac{1}{2} \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^3} \left(\frac{H}{H_c}\right)^2 \sum_{s=-1}^{+1} \sum_{n=0}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}\left(\frac{p_z}{mc}\right) \frac{mc^2 (2n+1+s)}{E(p_z, n, s)} F(p_z, n, s) \tag{36}$$

$$P_\parallel = \frac{1}{2} \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^3} \left(\frac{H}{H_c}\right) \sum_{s=-1}^{+1} \sum_{n=0}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}\left(\frac{p_z}{mc}\right) \frac{c^2 p_z^2}{E(p_z, n, s)} F(p_z, n, s) \,. \tag{37}$$

Rewriting Equation (19) in the form ($x \equiv p_z/mc$):

$$E(x, n, r) = \sqrt{1 + x^2 + 2 \frac{H}{H_c} (n + r - 1)} \qquad s = 2r - 3 \,, \tag{38}$$

we finally obtain:

$$P_\perp = \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^3} \left(\frac{H}{H_c}\right)^2 \sum_{r=1,2} \sum_{n=0}^{\infty} \int\limits_{-\infty}^{\infty} \mathrm{d}x \frac{n + r - 1}{E(x, n, r)} F(x, n, r) \tag{39}$$

$$P_\parallel = \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^3} \frac{H}{H_c} \sum_{r=1,2} \sum_{n=0} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \frac{x^2}{E(x, n, r)} F(x, n, r) \,, \tag{40}$$

where ($\beta = \tilde{\beta} mc^2$, $\mu = \tilde{\mu}/mc^2$)

$$F(x, n, r) = \{1 + \exp\beta [E(x, n, r) - \mu]\}^{-1} \,. \tag{41}$$

Equations (39) and (40) are identical with those obtained in CANUTO and CHIU (1968a), using the energy-momentum tensor and Dirac wave functions. In CANUTO and CHIU (1968a) we have also shown that Equations (39) and (40) reduce to the well-known expressions for a Fermi gas when most electrons are in states of quantum number $n \gg 1$. The thermodynamic properties of a magnetized Fermi gas (MFG) have been extensively studied in CANUTO and CHIU (1968b) as well as the magnetic moment of the system (CANUTO and CHIU, 1968c). Finally the energy and particle densities are trivially obtained using the same method of taking statistical averages, and they are:

$$\begin{aligned} U &= \text{energy density} \\ &= \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^2} \frac{H}{H_c} \sum_{r=1,2} \sum_{n=0} \int_{-\infty}^{+\infty} E(x, n, r)\, F(x, n, r)\, \mathrm{d}x \end{aligned} \tag{42}$$

$$\begin{aligned} N &= \text{particle density} \\ &= \frac{1}{4\pi^2} \frac{1}{\lambda_c^3} \frac{H}{H_c} \sum_{r=1,2} \sum_{n=0} \int_{-\infty}^{+\infty} F(x, n, r)\, \mathrm{d}x \end{aligned} \tag{43}$$

Equations (39), (40), (42), and (43) constitute the complete set of equations of state of an MFG.

4. The Anomalous Magnetic Moment

The eigenvalues given by Equation (19) do not include the anomalous magnetic moment of the electron, which has been shown to be

$$\mu = \mu_B (1 + \alpha/2\pi) \quad (\alpha = 1/137). \tag{44}$$

Solutions to the Dirac equation with an anomalous magnetic moment have been obtained by TERNOV *et al.* (1966), and we have studied the properties of an electron gas with an anomalous magnetic moment in a magnetic field (CHIU *et al.*, 1968). We have found that the thermodynamic properties of an electron gas (e.g., pair-creation equilibrium) are substantially altered when the magnetic field H is of the order or exceeds $(4\pi/\alpha)\, H_c \cong 10^{16}$ gauss. It is not known whether magnetic fields as strong as that can exist in nature. Further, the solution of TERNOV *et al.* (1966) may not be applicable to field strengths of such a magnitude, since their solution is based on the concept of a point magnetic moment, while electrodynamics predicts that the anomalous magnetic moment has a form factor of quantum-electrodynamical nature.

5. Astrophysical Applications

Generally one does not expect fields stronger than 10^{14} gauss to be achieved in collapsed objects. In order to have a field of 10^{16} gauss, for example, in a neutron star, the initial field must exceed 10^6 gauss (assuming a scale factor of 10^5 in a collapse process), while the maximum field observed is of the order of 3×10^4 gauss. In white dwarfs one generally cannot expect a field strength exceeding 10^9 or 10^{10} gauss. As shown previously, a magnetized gas will behave as a classical gas when states of large quantum numbers are occupied. Thus, it is expected that even in the most strongly magnetized objects only the envelope will be altered by the presence of a magnetic field. However, in condensed objects like the white dwarf or the neutron stars the properties of a thin envelope determines almost all observable properties (e.g., luminosity, pulsation, etc.).

6. Discussion and Conclusion

In this paper we have rederived the equations of state of a magnetized Fermi gas. The ordinary quantum-mechanical commutation relations are not valid in a magnetic field. The usual commutation relation $[v_x, v_y]=0$ is no longer valid. This is because the magnetic field exerts a force on a charged particle. Although the magnetic force, being perpendicular to the direction of motion, cannot impart or extract energy from the particle, it does alter the direction of motion of the particle, causing the particle to move in circular orbits. The two components of the velocity of the particle perpendicular to the magnetic field thus become correlated. In fact, a suitable transformation (13) (not a canonical transformation) transforms the Hamiltonian of the particle (in the non-relativistic approximation) into that of a simple harmonic oscillator. From this transformed Hamiltonian we thus readily obtain the energy eigenvalues of a non-relativistic electron in a magnetic field. An obvious relativistic generalization of the non-relativistic energy eigenvalues gives the correct expression for the relativistic case.

Using the relativistic energy eigenvalues, new velocities redefined in Equations (5) and (13), and the usual thermodynamic definition of pressure, we obtained the stress-energy tensor for a magnetized Fermi gas. The internal energy and particle density are trivially obtained from simple considerations. The new velocities defined in Equations (5) and (13), with proper relativistic generalizations, should be useful in obtaining transport properties in a magnetized Fermi gas.

"Del senno di poi son piene le fosse" ("Hindsight is easier than foresight") (MANZONI, 1827).

Acknowledgements

We would like to thank Professor Edwin E. Salpeter for discussions on the anomalous magnetic moment. One of us (V.C.) acknowledges a National Academy of Sciences-National Research Council resident research associateship.

References

CANUTO, V. and CHIU, H. Y.: 1968a, 'Quantum Theory of an Electron Gas in Intense Magnetic Fields', *Phys. Rev.* **173**, 1210.

CANUTO, V. and CHIU, H. Y.: 1968b, 'Thermodynamic Properties of a Magnetized Fermi Gas', *Phys. Rev.* **173**, 1220.

Canuto, V. and Chiu, H. Y.: 1968c, 'The Magnetic Moment of a Magnetized Fermi Gas', *Phys. Rev.* **173,** 1229.

Chiu, H. Y. and Canuto, V.: 1968, *Phys. Rev. Letters* **21,** No. 5, 110.

Chiu, H. Y., Canuto, V., and Fassio-Canuto, L.: 1968, 'Quantum Theory of an Electron Gas with Anomalous Magnetic Moments in Intense Magnetic Fields', *Phys. Rev.* (in press).

Kubo, R.: 1965, in *Statistical Mechanics*, North-Holland Publ. Co., Amsterdam, pp. 278–280.

Landau, L. D.: 1930, *Z. Phys.* **64,** 629. (In *Collected Papers of L.D. Landau*, Gordon and Breach Science Publishers, New York, 1967, pp. 31–38.)

Landau, L. D. and Lifschitz, E. M.: 1958, in *Quantum Mechanics*, Pergamon Press Ltd., Reading, Mass., pp. 471–487.

Landau, L. D. and Lifshitz, E. M.: 1962, in *The Classical Theory of Fields*, Pergamon Press, Reading, Mass., pp. 97–99.

Manzoni, A.: 1827, in *I Promessi Sposi*, Milano (revised ed.: 1840). Translation available in many languages.

Newton, T. D. and Wigner, E. P.: 1949, *Rev. Mod. Phys.* **21,** 400.

Sakurai, J. J.: 1967, in *Advanced Quantum Mechanics*, Addison-Wesley Publ. Co., Reading, Mass., pp. 117–121.

Schwinger, J.: 1948, *Phys. Rev.* **73,** 416.

Ternov, I. M., Bagzov, V. G., and Zhukovskii, V., Ch.: 1966, *Moscow Univ. Bull.* **21,** No. 1, 21.